ON A CLASS OF FULLY NONLINEAR ELLIPTIC EQUATIONS ON HERMITIAN MANIFOLDS

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Abstract. We derive *a priori* C^2 estimates for a class of complex Monge-Ampère type equations on Hermitian manifolds. As an application we solve the Dirichlet problem for these equations under the assumption of existence of a subsolution; the existence result, as well as the second order boundary estimates, is new even for bounded domains in $\mathbb{C}^n$.

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1. Introduction

Let (M^n, ω) be a compact Hermitian manifold of dimension $n \geq 2$ with smooth boundary ∂M and χ a smooth real $(1,1)$ form on $\bar{M} := M \cup \partial M$. Define for a function $u \in C^2(M)$,

$$\chi_u = \chi + \frac{\sqrt{-1}}{2}\partial\bar{\partial}u$$

and set

$$[\chi] = \big\{\chi_u : u \in C^2(M)\big\}, \quad [\chi]^+ = \big\{\chi' \in [\chi] : \chi' > 0\big\}.$$

In this paper we are concerned with the equation for $1 \leq \alpha \leq n$,

$$\chi_u^n = \psi \chi_u^{n-\alpha} \wedge \omega^\alpha \;\text{ in } M, \; . \tag{1.1}$$

We require $\chi_u > 0$ so that equation (1.1) is elliptic; we call such functions *admissible* or χ-*plurisubharmonic*. Consequently, we assume $\psi > 0$ on $\bar{M}$; equation (1.1) becomes degenerate when $\psi \geq 0$.

When $\alpha = n$ this is the complex Monge-Ampère equation which plays extremely important roles in complex geometry and analysis, especially in Kähler geometry, and has received extensive study since the fundamental work of Yau [34] (see also [1]) on compact Kähler manifolds and that of Caffarelli, Kohn, Nirenberg and Spruck [3] for the Dirichlet problem in strongly pseudoconvex domains in $\mathbb{C}^n$. For $\alpha = 1$ equation (1.1) also arises naturally in geometric problems; it was posed by Donaldson [11]

in connection with moment maps and is closely related to the Mabuchi energy [5], [33], [28].

Donaldson's problem assumes M is closed, both ω, χ are Kähler and ψ is constant. It was studied by Chen [5], Weinkove [32], [33], Song and Weinkove [28] using parabolic methods. In [28] Song and Weinkove give a necessary and sufficient solvability condition. Their result was extended by Fang, Lai and Ma [12] to all $1 \leq \alpha < n$.

In this paper we study the Dirichlet problem for equation (1.1) on Hermitian manifolds. Given $\psi \in C^\infty(\bar{M})$ and $\varphi \in C^\infty(\partial M)$, we wish to find a solution $u \in C^\infty(\bar{M})$ of equation (1.1) satisfying the boundary condition

$$u = \varphi \quad \text{on } \partial M. \tag{1.2}$$

The Dirichlet problem for the complex Monge-Ampère equation in $\mathbb{C}^n$ was studied by Caffarelli, Kohn, Nirenberg and Spruck [3] on strongly pseudoconvex domains. Their result was extended to Hermitian manifolds by Cherrier and Hanani [8], [23], and by the first author [14] to arbitrary bounded domains in $\mathbb{C}^n$ under the assumption of existence of a subsolution. See also the more recent papers [16], [35], and related work of Tosatti and Weinkove [29], [30] who completely extended the zero order estimate of Yau [34] on closed Kähler manifolds to the Herimatian case. In [25] Li treated the Dirichlet problem for more general fully nonlinear elliptic equations in $\mathbb{C}^n$ but needed to assume the existence of a *strict* subsolution. Li's result does not cover equation (1.1) as it fails to satisfy some of the key structure conditions in [25].

In this paper we prove the following existence result which is new even in the case when M is a bounded domain in $\mathbb{C}^n$ and $\chi = 0$; we assume $2 \leq \alpha \leq n-2$ as the cases $\alpha = 1$ and $\alpha = n-1$ were considered in [17] and [18], while for the complex Monge-Ampère equation ($\alpha = n$) it was proved in [16].

Theorem 1.1. *Let $\psi \in C^\infty(\bar{M})$, $\psi > 0$ and $\varphi \in C^\infty(\partial M)$. There exists a unique admissible solution $u \in C^\infty(\bar{M})$ of the Dirichlet problem* (1.1)-(1.2)*, provided that there exists an admissible subsolution $\underline{u} \in C^2(\bar{M})$:*

$$\begin{cases} \chi_{\underline{u}}^n \geq \psi \chi_{\underline{u}}^{n-\alpha} \wedge \omega^\alpha & \text{on } \bar{M} \\ \underline{u} = \varphi & \text{on } \partial M. \end{cases} \tag{1.3}$$

In order to solve the Dirichlet problem (1.1)-(1.2) one needs to derive *a priori* C^2 estimates up to the boundary for admissible solutions. The most difficult step is probably the second order estimates on the boundary.

Theorem 1.2. *Suppose $\psi \in C^1(\bar{M})$, $\psi > 0$ and $\varphi \in C^4(\partial M)$ and $\underline{u} \in C^2(\bar{M})$ is an admissible subsolution satisfying* (1.3). *Let $u \in C^3(\bar{M})$ be an admissible solution of the Dirichlet problem* (1.1)-(1.2). *Then*

$$\max_{\partial M} |\nabla^2 u| \leq C \tag{1.4}$$

where C depends on $|u|_{C^1(\bar{M})}$, $\min \psi^{-1}$, $|\underline{u}|_{C^2(\bar{M})}$ and $\min\{c_1 : c_1\chi_{\underline{u}} \geq \omega\}$, as well as other known data.

This estimate is new for domains in $\mathbb{C}^n$. Note that ∂M is assumed to be smooth and compact in Theorem 1.2, but otherwise is completely arbitrary. In general, the Dirichlet problem (1.1)-(1.2) is not always solvable in an arbitrary smooth bounded domain in $\mathbb{C}^n$ without the subsolution assumption. In the theory of nonlinear elliptic equations, many well known classical results assume certain geometric conditions on the boundary of the underlying domain; see e.g. [27], [3], [2] and [4]. In [19], [13] and [14], J. Spruck and the first author were able to solve the Dirichlet problem for real and complex Monge-Ampère equations on arbitrary smooth bounded domains assuming the existence of a subsolution. Their work was motivated by applications to geometric problems and had been found useful in some important problems such as the proof by P.-F. Guan [20], [21] of the Chern-Levine-Nirenberg conjecture [6], and work on the Donaldson conjectures [10] on geodesics in the space of Kähler metrics; we refer the reader to [26] for recent progress and further references on this fast-developing subject.

On a closed Kähler manifold (M, ω), Fang, Lai and Ma [12] proved second and zero order estimates for equation (1.1) when χ is also Kähler and ψ is constant. We extend their second order estimates to Hermitian manifolds and for general χ and ψ. Technically the major difficulty is to control extra third order terms which occur due to the nontrivial torsion of the Hermitian metric. This was done in [17], [18] for $\alpha = 1$ and $\alpha = n - 1$; the case $2 \leq \alpha \leq n - 2$ is considerably more complicated. In order to solve the Dirichlet problem we also need global gradient estimates. Following [28] and [12] let

$$\mathscr{C}_\alpha(\omega) = \big\{[\chi] : \exists\, \chi' \in [\chi]^+,\ n\chi'^{n-1} > (n-\alpha)\psi\chi'^{n-\alpha-1} \wedge \omega^\alpha\big\}. \tag{1.5}$$

Theorem 1.3. *Let $u \in C^4(M) \cap C^2(\bar{M})$ be an admissible solution of equation (1.1) where $\psi \in C^2(\bar{M})$, $\psi > 0$. Suppose that $\chi \in \mathscr{C}_\alpha(\omega)$. Then there are constants C_1, C_2*

depending on $|u|_{C^0(\bar{M})}$ *such that*

$$\sup_M |\nabla u| \leq C_1(1 + \sup_{\partial M} |\nabla u|), \tag{1.6}$$

$$\sup_M \Delta u \leq C_2(1 + \sup_{\partial M} \Delta u). \tag{1.7}$$

In particular, if M *is closed* ($\partial M = \emptyset$) *then* $|\nabla u| \leq C_1$ *and* $|\Delta u| \leq C_2$ *on* M.

The cone $\mathscr{C}_\alpha(\omega)$ was first introduced by Song and Weinkove [28] ($\alpha = 1$) and Fang, Lai and Ma [12] who derived the estimate (1.7) on a closed Kähler manifold (M, ω) when χ is also Kähler and

$$\psi = c_\alpha := \frac{\int_M \chi^n}{\int_M \chi^{n-\alpha} \wedge \omega^\alpha},$$

which is a Kähler class invariant. As in [28], [12] the constant C_2 in Theorem 1.3 is independent of gradient bounds, i.e. C_2 is independent of C_1.

The subsolution assumption (1.3) implies $[\chi] \in \mathscr{C}_\alpha(\omega)$. On a closed manifold, a subsolution must be a solution or the equation has no solution. This is a consequence of the maximum principle and a concavity property of equation (1.1).

The gradient estimate (1.6) is crucial to the proof of Theorem 1.1 and is also new when ω and χ are Kähler. Indeed, deriving gradient estimates for fully nonlinear equations on complex manifolds turns out to be a rather challenging and mostly open question. Only very recently were Dinew and Kolodziej [9] able to prove the gradient estimate using scaling techniques and Liouville type theorems for the complex Hessian equation

$$\omega^n = \psi \chi_u^{n-\alpha} \wedge \omega^\alpha \tag{1.8}$$

on closed Kähler manifolds which is consequently solvable due to the earlier work of Hou, Ma and Wu [24].

The proof of Theorem 1.3 is carried out in Sections 3 and 5 where we derive the estimates for $|\nabla u|$ and Δu, the gradient and Laplacian of u, respectively. In Section 4 we establish the boundary estimates for second derivatives. These estimates allow us to derive global estimates for all (real) second derivatives as in Section 5 in [16] and apply the Evans-Krylov theorem since equation (1.1) becomes uniformly elliptic. Theorem 1.1 may then be proved by the continuity method. These steps are all well understood so we shall omit them. In section 2 we recall some formulas on Hermitian manifolds.

2. Preliminaries

Let g and ∇ denote the Riemannian metric and Chern connection of (M,ω). The torsion and curvature tensors of ∇ are defined by

$$\begin{aligned} T(u,v) &= \nabla_u v - \nabla_v u - [u,v], \\ R(u,v)w &= \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w, \end{aligned} \tag{2.1}$$

respectively. Following the notations in [16], in local coordinates $z = (z_1,\dots,z_n)$ we have

$$\begin{cases} g_{i\bar j} &= g\Big(\dfrac{\partial}{\partial z_i}, \dfrac{\partial}{\partial \bar z_j}\Big), \quad \{g^{i\bar j}\} = \{g_{i\bar j}\}^{-1}, \\ T^k_{ij} &= \Gamma^k_{ij} - \Gamma^k_{ji} = g^{k\bar l}\Big(\dfrac{\partial g_{j\bar l}}{\partial z_i} - \dfrac{\partial g_{i\bar l}}{\partial z_j}\Big), \\ R_{i\bar j k\bar l} &= -g_{m\bar l}\dfrac{\partial \Gamma^m_{ik}}{\partial \bar z_j} = -\dfrac{\partial^2 g_{k\bar l}}{\partial z_i \partial \bar z_j} + g^{p\bar q}\dfrac{\partial g_{k\bar q}}{\partial z_i}\dfrac{\partial g_{p\bar l}}{\partial \bar z_j}. \end{cases} \tag{2.2}$$

Recall that for a smooth function v, $v_{i\bar j} = v_{\bar j i} = \partial_i\bar\partial_j v$, $v_{i\bar j k} = \partial_k v_{i\bar j} - \Gamma^l_{ki}v_{l\bar j}$ and

$$v_{i\bar j k\bar l} = \bar\partial_l v_{i\bar j k} - \overline{\Gamma^q_{lj}}v_{i\bar q k}.$$

We have (see e.g. [17]),

$$\begin{cases} v_{i\bar j k} - v_{k\bar j i} = T^l_{ik}v_{l\bar j}, \\ v_{i\bar j\bar k} - v_{i\bar k\bar j} = \overline{T^l_{jk}}v_{i\bar l}, \end{cases} \tag{2.3}$$

$$\begin{cases} v_{i\bar j k\bar l} - v_{i\bar j\bar l k} = g^{p\bar q}R_{k\bar l i\bar q}v_{p\bar j} - g^{p\bar q}R_{k\bar l p\bar j}v_{i\bar q}, \\ v_{i\bar j k\bar l} - v_{k\bar l i\bar j} = g^{p\bar q}(R_{k\bar l i\bar q}v_{p\bar j} - R_{i\bar j k\bar q}v_{p\bar l}) + T^p_{ik}v_{p\bar j\bar l} + \overline{T^q_{jl}}v_{i\bar q k} - T^p_{ik}\overline{T^q_{jl}}v_{p\bar q}. \end{cases} \tag{2.4}$$

Let $u \in C^4(M)$ be an admissible solution of equation (1.1). As in [16] and [17], we denote $\mathfrak{g}_{i\bar j} = \chi_{i\bar j} + u_{i\bar j}$, $\{\mathfrak{g}^{i\bar j}\} = \{\mathfrak{g}_{i\bar j}\}^{-1}$ and $w = \mathrm{tr}\chi + \Delta u$. Note that $\{\mathfrak{g}_{i\bar j}\}$ is positive definite. Assume at a fixed point $p \in M$ that $g_{i\bar j} = \delta_{ij}$ and $\mathfrak{g}_{i\bar j}$ is diagonal. Then

$$u_{i\bar i k\bar k} - u_{k\bar k i\bar i} = R_{k\bar k i\bar p}u_{p\bar i} - R_{i\bar i k\bar p}u_{p\bar k} + 2\mathfrak{Re}\{\overline{T^j_{ik}}u_{i\bar j k}\} - T^p_{ik}\overline{T^q_{ik}}u_{p\bar q}, \tag{2.5}$$

and therefore,

$$\mathfrak{g}_{i\bar i k\bar k} - \mathfrak{g}_{k\bar k i\bar i} = R_{k\bar k i\bar i}\mathfrak{g}_{i\bar i} - R_{i\bar i k\bar k}\mathfrak{g}_{k\bar k} + 2\mathfrak{Re}\{\overline{T^j_{ik}}\mathfrak{g}_{i\bar j k}\} - |T^j_{ik}|^2\mathfrak{g}_{j\bar j} - G_{i\bar i k\bar k} \tag{2.6}$$

where

$$G_{i\bar i k\bar k} = \chi_{k\bar k i\bar i} - \chi_{i\bar i k\bar k} + R_{k\bar k i\bar p}\chi_{p\bar i} - R_{i\bar i k\bar p}\chi_{p\bar k} + 2\mathfrak{Re}\{\overline{T^j_{ik}}\chi_{i\bar j k}\} - T^p_{ik}\overline{T^q_{ik}}\chi_{p\bar q}. \tag{2.7}$$

Let $S_k(\lambda)$ denote the k-th elementary symmetric polynomial of $\lambda \in \mathbb{R}^n$

$$S_k(\lambda) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}.$$

In local coordinates we can write equation (1.1) in the form

$$F(\mathfrak{g}_{i\bar{j}}) := \Big(\frac{S_n(\lambda_*(\mathfrak{g}_{i\bar{j}}))}{S_{n-\alpha}(\lambda_*(\mathfrak{g}_{i\bar{j}}))}\Big)^{\frac{1}{\alpha}} = \Big(\frac{\psi}{C_n^\alpha}\Big)^{\frac{1}{\alpha}} \tag{2.8}$$

or equivalently,

$$C_n^\alpha \psi^{-1} = S_\alpha(\lambda^*(\mathfrak{g}^{i\bar{j}})) \tag{2.9}$$

where $\lambda_*(A)$ and $\lambda^*(A)$ denote the eigenvalues of a Hermitian matrix A with respect to $\{g_{i\bar{j}}\}$ and to $\{g^{i\bar{j}}\}$, respectively. Unless otherwise indicated we shall use S_α to denote $S_\alpha(\lambda^*(\mathfrak{g}^{i\bar{j}}))$ when no possible confusion would occur. We shall also occasionally write $F(\chi_u) := F(\mathfrak{g}_{i\bar{j}})$ and $F(\chi_{\underline{u}}) := F(\underline{u}_{i\bar{j}} + \chi_{i\bar{j}})$, etc.

Differentiating equation (2.9) twice at a point p where $g_{i\bar{j}} = \delta_{ij}$ and $\mathfrak{g}_{i\bar{j}}$ is diagonal, we obtain

$$C_n^\alpha \partial_l(\psi^{-1}) = -\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2\mathfrak{g}_{i\bar{i}l} \tag{2.10}$$

and

$$\begin{aligned} C_n^\alpha \bar{\partial}_l\partial_l(\psi^{-1}) = & -\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2\mathfrak{g}_{i\bar{i}l\bar{l}} + \sum_{i,j} S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2\mathfrak{g}^{j\bar{j}}(\mathfrak{g}_{i\bar{j}l}\mathfrak{g}_{j\bar{i}\bar{l}} + \mathfrak{g}_{j\bar{i}l}\mathfrak{g}_{i\bar{j}\bar{l}}) \\ & + \sum_{i\neq j} S_{\alpha-2;ij}(\mathfrak{g}^{i\bar{i}})^2(\mathfrak{g}^{j\bar{j}})^2\big(\mathfrak{g}_{i\bar{i}l}\mathfrak{g}_{j\bar{j}\bar{l}} - \mathfrak{g}_{j\bar{i}l}\mathfrak{g}_{i\bar{j}\bar{l}}\big) \end{aligned} \tag{2.11}$$

where for $\{i_1,\cdots,i_s\} \subseteq \{1,\cdots,n\}$,

$$S_{k;i_1\cdots i_s}(\lambda) = S_k(\lambda|_{\lambda_{i_1}=\cdots=\lambda_{i_s}=0}).$$

We need the following inequality from [22]; see also Proposition 2.2 in [12],

$$\sum_{i=1}^n \frac{S_{\alpha-1;i}(\lambda)}{\lambda_i}\xi_i\bar{\xi}_i + \sum_{i,j} S_{\alpha-2;ij}(\lambda)\xi_i\bar{\xi}_j \geq \sum_{i,j}\frac{S_{\alpha-1;i}(\lambda)S_{\alpha-1;j}(\lambda)}{S_\alpha(\lambda)}\xi_i\bar{\xi}_j \geq 0 \tag{2.12}$$

for $\lambda = (\lambda_1,\ldots,\lambda_n)$, $\lambda_i > 0$ and $(\xi_1,\ldots,\xi_n) \in \mathbb{C}^n$. Apply (2.12) to $\lambda_i = \mathfrak{g}^{i\bar{i}}$, $\xi_i = (\mathfrak{g}^{i\bar{i}})^2\mathfrak{g}_{i\bar{i}l}$ and sum over l. We see that

$$\sum_{i,l} S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^3\mathfrak{g}_{i\bar{i}l}\mathfrak{g}_{i\bar{i}\bar{l}} + \sum_{i\neq j}\sum_l S_{\alpha-2;ij}(\mathfrak{g}^{i\bar{i}})^2(\mathfrak{g}^{j\bar{j}})^2\mathfrak{g}_{i\bar{i}l}\mathfrak{g}_{j\bar{j}\bar{l}} \geq 0. \tag{2.13}$$

Note also that

$$\sum_{i\neq j}(S_{\alpha-1;i} - S_{\alpha-2;ij}\mathfrak{g}^{j\bar{j}})(\mathfrak{g}^{i\bar{i}})^2\mathfrak{g}^{j\bar{j}}\mathfrak{g}_{j\bar{i}l}\mathfrak{g}_{i\bar{j}\bar{l}} \geq 0.$$

We obtain from (2.11),

$$\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2\mathfrak{g}_{i\bar{i}l\bar{l}} \geq \sum_{i,j} S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2\mathfrak{g}^{j\bar{j}}\mathfrak{g}_{i\bar{j}l}\mathfrak{g}_{j\bar{i}\bar{l}} - C. \tag{2.14}$$

Let $\underline{u} \in C^2(\bar{M})$, $\chi_{\underline{u}} > 0$ such that

$$n\chi_{\underline{u}}^{n-1} > (n-\alpha)\psi\chi_{\underline{u}}^{n-\alpha-1} \wedge \omega^\alpha. \tag{2.15}$$

Thus there is $\epsilon > 0$ such that

$$\epsilon\omega \leq \chi_{\underline{u}} \leq \epsilon^{-1}\omega. \tag{2.16}$$

The key ingredient of our estimates in the following sections is the following lemma.

Lemma 2.1. *There exist constants $N, \theta > 0$ such that when $w \geq N$ at a point p where $g_{i\bar{j}} = \delta_{ij}$ and $\mathfrak{g}_{i\bar{j}}$ is diagonal,*

$$\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2(\underline{u}_{i\bar{i}} - u_{i\bar{i}}) \geq \theta\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2 + \theta \tag{2.17}$$

and, equivalently,

$$\sum_{i,j} F^{i\bar{j}}(\underline{u}_{i\bar{j}} - u_{i\bar{j}}) \geq \theta\sum_{i,j} F^{i\bar{j}}g_{i\bar{j}} + \theta. \tag{2.18}$$

Here and in the rest of this paper,

$$F^{i\bar{j}} = \frac{\partial F}{\partial\mathfrak{g}_{i\bar{j}}}(\mathfrak{g}_{i\bar{j}}).$$

It is well known that $\{F^{i\bar{j}}\}$ is positive definite.

An equivalent form of Lemma 2.1 and its proof are given in [12] (Theorem 2.8); see also [15] where it is proved for more general fully nonlinear equations. So we shall omit the proof here.

3. The gradient estimates

In this section we establish the *a priori* gradient estimates.

Proposition 3.1. *Suppose $\chi \in \mathscr{C}_\alpha(\omega)$ and let $u \in C^3(M) \cap C^1(\bar{M})$ be an admissible solution of* (1.1). *There is a uniform constant $C > 0$ such that*

$$\sup_{\bar{M}} |\nabla u| \leq C(1 + \sup_{\partial M} |\nabla u|). \tag{3.1}$$

Proof. Let $\underline{u} \in C^2(\bar{M})$, $\chi_{\underline{u}} > 0$ satisfy (2.15) and consider $\phi = Ae^\eta$ where

$$\eta = \underline{u} - u + \sup_M (u - \underline{u})$$

and A is a constant to be determined. Suppose the function $e^\phi |\nabla u|^2$ attains its maximal value at an interior point $p \in M$. Choose local coordinate around p such that $g_{i\bar{j}} = \delta_{ij}$ and $\mathfrak{g}_{i\bar{j}}$ is diagonal at p. At p we have

$$\frac{\partial_i(|\nabla u|^2)}{|\nabla u|^2} + \partial_i \phi = 0, \quad \frac{\bar{\partial}_i(|\nabla u|^2)}{|\nabla u|^2} + \bar{\partial}_i \phi = 0 \tag{3.2}$$

and

$$\frac{\bar{\partial}_i \partial_i(|\nabla u|^2)}{|\nabla u|^2} - \frac{\partial_i(|\nabla u|^2)\bar{\partial}_i(|\nabla u|^2)}{|\nabla u|^4} + \bar{\partial}_i \partial_i \phi \leq 0. \tag{3.3}$$

By direct computation,

$$\partial_i(|\nabla u|^2) = \sum_k (u_k u_{i\bar{k}} + u_{ki} u_{\bar{k}}), \tag{3.4}$$

$$\begin{aligned} \bar{\partial}_i \partial_i(|\nabla u|^2) &= \sum_k (u_{k\bar{i}} u_{\bar{k}i} + u_{ki} u_{\bar{k}\bar{i}} + u_{ki\bar{i}} u_{\bar{k}} + u_k u_{\bar{k}i\bar{i}}) \\ &= \sum_k (u_{ki} u_{\bar{k}\bar{i}} + u_{i\bar{i}k} u_{\bar{k}} + u_{i\bar{i}\bar{k}} u_k + R_{i\bar{i}k\bar{l}} u_l u_{\bar{k}}) \\ &\quad + \sum_k \Big| u_{\bar{k}i} - \sum_l T^k_{il} u_{\bar{l}} \Big|^2 - \sum_k \Big| \sum_l T^k_{il} u_{\bar{l}} \Big|^2. \end{aligned} \tag{3.5}$$

Therefore, by (2.3) and (2.10),

$$\begin{aligned} \sum_i S_{\alpha-1;i} (\mathfrak{g}^{i\bar{i}})^2 \bar{\partial}_i \partial_i (|\nabla u|^2) \geq{}& \sum_{i,k} S_{\alpha-1;i} (\mathfrak{g}^{i\bar{i}})^2 |u_{ki}|^2 \\ & - C|\nabla u|^2 - C|\nabla u|^2 \sum_i S_{\alpha-1;i} (\mathfrak{g}^{i\bar{i}})^2. \end{aligned} \tag{3.6}$$

From (3.2) and (3.4),

$$\begin{aligned}\big|\partial_i(|\nabla u|^2)\big|^2 &= \Big|\sum_k u_{ki}u_{\bar k}\Big|^2 - 2|\nabla u|^2\sum_k \mathfrak{Re}\big\{u_k u_{i\bar k}\phi_{\bar i}\big\} - \Big|\sum_k u_k u_{i\bar k}\Big|^2 \\ &\leq |\nabla u|^2\sum_k |u_{ki}|^2 - 2|\nabla u|^2\sum_k \mathfrak{Re}\big\{u_k u_{i\bar k}\phi_{\bar i}\big\}\end{aligned} \tag{3.7}$$

by Schwarz inequality.

Combining (3.3), (3.6) and (3.7) we derive

$$\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar i})^2(\phi_{i\bar i} - C) + \frac{2}{|\nabla u|^2}\sum_{i,k} S_{\alpha-1;i}(\mathfrak{g}^{i\bar i})^2\mathfrak{Re}\big\{u_k u_{i\bar k}\phi_{\bar i}\big\} \leq C. \tag{3.8}$$

Next,

$$\partial_i\phi = \phi\,\partial_i\eta, \quad \bar\partial_i\partial_i\phi = \phi\,(|\partial_i\eta|^2 + \bar\partial_i\partial_i\eta).$$

Therefore,

$$2\phi^{-1}\sum_k \mathfrak{Re}\big\{u_k u_{i\bar k}\phi_{\bar i}\big\} \geq 2\mathfrak{g}_{i\bar i}\mathfrak{Re}\big\{u_i\eta_{\bar i}\big\} - \frac{1}{2}|\nabla u|^2|\eta_i|^2 - C \tag{3.9}$$

and

$$\begin{aligned}\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar i})^2\eta_{i\bar i} &+ \frac{1}{2}\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar i})^2|\eta_i|^2 \\ &\leq -\frac{2}{|\nabla u|^2}\sum_i S_{\alpha-1;i}\mathfrak{g}^{i\bar i}\mathfrak{Re}\big\{u_i\eta_{\bar i}\big\} + \frac{C}{\phi} \\ &+C\Big(\frac{1}{\phi} + \frac{1}{|\nabla u|^2}\Big)\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar i})^2.\end{aligned} \tag{3.10}$$

For $N > 0$ sufficiently large so that Lemma 2.1 holds, we consider two cases: (**a**) $w > N$ and (**b**) $w \leq N$. Without loss of generality we can assume that $|\nabla u| > |\nabla \underline{u}|$ at p or otherwise we are done. Note that

$$-\frac{2}{|\nabla u|^2}\sum_i S_{\alpha-1;i}\mathfrak{g}^{i\bar i}\mathfrak{Re}\big\{u_i\eta_{\bar i}\big\} \leq 4\sum_i S_{\alpha-1;i}\mathfrak{g}^{i\bar i} = 4\alpha S_\alpha. \tag{3.11}$$

In case (**a**) we have by Lemma 2.1

$$\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar i})^2\eta_{i\bar i} \geq \theta + \theta\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar i})^2. \tag{3.12}$$

So if $S_{\alpha-1;i}(\mathfrak{g}^{i\bar i})^2 \geq K$ for some i and K sufficiently large we derive a bound $|\nabla u| \leq C$ from (3.10) and (3.11) when A is sufficiently large.

Suppose that $S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2 \leq K$ for all i and assume $\mathfrak{g}_{1\bar{1}} \leq \cdots \leq \mathfrak{g}_{n\bar{n}}$. Note that

$$\prod_{i=1}^{\alpha} \mathfrak{g}^{i\bar{i}} \geq \frac{S_\alpha}{C_n^\alpha} = \frac{1}{\psi}.$$

We have

$$\frac{\mathfrak{g}^{1\bar{1}}}{\psi} \leq (\mathfrak{g}^{1\bar{1}})^2 \prod_{i=2}^{\alpha} \mathfrak{g}^{i\bar{i}} \leq S_{\alpha-1;1}(\mathfrak{g}^{1\bar{1}})^2 \leq K.$$

Therefore, for all $1 \leq i \leq n$,

$$S_{\alpha-1;i} \leq C_n^{\alpha-1}(\mathfrak{g}^{1\bar{1}})^{\alpha-1} \leq C_n^{\alpha-1}(K\psi)^{\alpha-1} \leq K'.$$

By Schwarz inequality,

$$\begin{aligned} -2\sum_i S_{\alpha-1;i}\mathfrak{g}^{i\bar{i}}\mathfrak{Re}\{u_i\eta_{\bar{i}}\} &\leq 4\sum_i S_{\alpha-1;i} + \frac{1}{4}|\nabla u|^2 \sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2|\eta_i|^2 \\ &\leq \frac{1}{4}|\nabla u|^2 \sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2|\eta_i|^2 + C. \end{aligned} \tag{3.13}$$

From (3.10), (3.12) and (3.13) we obtain

$$\frac{\theta}{C} - \frac{1}{\phi} - \frac{1}{|\nabla u|^2} \leq 0.$$

This gives a bound for $|\nabla u|$ when A is chosen sufficiently large.

In case (**b**) we have

$$\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2|\eta_i|^2 \geq |\nabla\eta|^2 \min_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2 \geq \frac{|\nabla\eta|^2}{w^{\alpha+1}} \geq \frac{|\nabla\eta|^2}{N^{\alpha+1}}. \tag{3.14}$$

Substituting this into (3.10), we derive from (3.11) and (2.16),

$$\frac{|\nabla\eta|^2}{2N^{\alpha+1}} \leq 5\alpha S_\alpha + \frac{C}{\phi} + \Big(\frac{C}{\phi} + \frac{C}{|\nabla u|^2} - \epsilon\Big)\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2. \tag{3.15}$$

This gives a bound $|\nabla u| \leq C$. $\square$

4. Boundary estimates for second derivatives

In this section we prove Theorem 1.2. Throughout this section we assume that φ is extended smoothly to $\bar{M}$ and that $\underline{u} \in C^2(\bar{M})$ is a subsoltuion satisfying (1.3). As in [16] and [18] we follow the idea of [19], [13], [14] to use $\underline{u} - u$ in construction of barrier functions.

To derive (1.4) let us consider a boundary point $0 \in \partial M$. We use coordinates around 0 such that $\frac{\partial}{\partial x_n}$ is the interior normal direction to ∂M at 0 and $g_{i\bar{j}}(0) = \delta_{ij}$. For convenience we set

$$t_{2k-1} = x_k,\ t_{2k} = y_k,\ 1 \leq k \leq n-1;\ t_{2n-1} = y_n,\ t_{2n} = x_n.$$

Since $u - \varphi = 0$ on ∂M, one derives

$$|u_{t_\alpha t_\beta}(0)| \leq C, \quad \alpha, \beta < 2n \tag{4.1}$$

where C depends on $|u|_{C^1(\bar{M})}$, $|\underline{u}|_{C^1(\bar{M})}$, and geometric quantities of ∂M.

To estimate $u_{t_\alpha x_n}(0)$ for $\alpha \leq 2n$, we shall employ a barrier function of the form

$$v = (u - \underline{u}) + t\sigma - T\sigma^2 \text{ in } \Omega_\delta = M \cap B_\delta \tag{4.2}$$

where t, T are positive constants to be determined, B_δ is the (geodesic) ball of radius δ centered at p, and σ is the distance function to ∂M. Note that σ is smooth in $M_{\delta_0} := \{z \in M : \sigma(z) < \delta_0\}$ for some $\delta_0 > 0$.

Lemma 4.1. *There exists $c_0 > 0$ such that for T sufficiently large and t, δ sufficiently small, $v \geq 0$ and*

$$\sum_{i,j} F^{i\bar{j}} v_{i\bar{j}} \leq -c_0\Big(1 + \sum_{i,j} F^{i\bar{j}} g_{i\bar{j}}\Big) \text{ in } \Omega_\delta. \tag{4.3}$$

Proof. The proof is very similar to that of Lemma 5.1 in [18]; for completeness we include it here. First of all, since σ is smooth and $\sigma = 0$ on ∂M, for fixed t and T we may require δ to be so small that $v \geq 0$ in Ω_δ. Next, note that

$$\sum_{i,j} F^{i\bar{j}} \sigma_{i\bar{j}} \leq C_1 \sum_{i,j} F^{i\bar{j}} g_{i\bar{j}}$$

for some constant $C_1 > 0$ under control. Therefore,

$$\sum_{i,j} F^{i\bar{j}} v_{i\bar{j}} \leq \sum_{i,j} F^{i\bar{j}} (u_{i\bar{j}} - \underline{u}_{i\bar{j}}) + C_1(t + T\sigma) \sum_{i,j} F^{i\bar{j}} g_{i\bar{j}} - 2T \sum_{i,j} F^{i\bar{j}} \sigma_i \sigma_{\bar{j}}. \tag{4.4}$$

Fix $N > 0$ sufficiently large so that Lemma 2.1 holds. At a fixed point in Ω_δ, we consider two cases: (a) $w \leq N$ and (b) $w > N$.

In case (a) let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\{\mathfrak{g}_{i\bar{j}}\}$. We see from equation (2.8) that there is a uniform lower bound $\lambda_1 \geq c_1 > 0$. Consequently, $c_2 I \leq \{F^{i\bar{j}}\} \leq \frac{1}{c_2} I$ for some constant $c_2 > 0$ depending on N and c_1, and hence

$$\sum_{i,j} F^{i\bar{j}} \sigma_i \sigma_{\bar{j}} \geq c_2 |\nabla \sigma|^2 = \frac{c_2}{4}. \tag{4.5}$$

Since F is homogeneous of degree one, by (4.4), (4.5) and (2.16),

$$\sum_{i,j} F^{i\bar{j}} v_{i\bar{j}} \leq F(\mathfrak{g}_{i\bar{j}}) + (C_1(t + T\sigma) - \epsilon) \sum_{i,j} F^{i\bar{j}} g_{i\bar{j}} - \frac{c_2 T}{2} \leq -\frac{\epsilon}{2}\Big(1 + \sum_{i,j} F^{i\bar{j}} g_{i\bar{j}}\Big) \tag{4.6}$$

if we fix T sufficiently large and require t and δ small to satisfy $C_1(t + T\delta) \leq \epsilon/2$.

Suppose now that $w > N$. By Lemma 2.1 and (4.4), we may further require t and δ to satisfy $C_1(t + T\delta) \leq \theta/2$ so that (4.3) holds. $\square$

Using Lemma 4.1 we may derive as in [16] (but see [18] for some corrections) the estimates $|u_{t_\alpha x_n}(0)| \leq C$ (and therefore $|u_{x_n t_\alpha}(0)| \leq C$) for $\alpha < 2n$; we shall omit the proof here. It remains to prove

$$\mathfrak{g}_{n\bar{n}}(0) \leq C. \tag{4.7}$$

The proof below uses an idea of Trudinger [31].

Let $T_C \partial M$ be the complex tangent bundle and

$$T^{1,0}\partial M = T^{1,0} M \bigcap T_C \partial M = \{\xi \in T^{1,0} M : d\sigma(\xi) = 0\}.$$

Let $\hat{\chi}_u$ and $\hat{\omega}$ denote the restrictions to $T_C \partial M$ of χ_u and ω respectively. As in [18] we only have to show that

$$m_0 := \min_{\partial M} \frac{n \hat{\chi}_u^{n-1}}{\psi (n - \alpha) \hat{\chi}_u^{n-\alpha-1} \wedge \hat{\omega}^\alpha} > 1.$$

Suppose that m_0 is reached at a point $0 \in \partial M$. Let $\tau_1, \cdots, \tau_{n-1}$ be a local frame of vector fields in $T_C^{1,0} \partial M$ around 0 such that $g(\tau_\beta, \bar{\tau}_\gamma) = \delta_{\beta\gamma}$ for $1 \leq \beta, \gamma \leq n - 1$ and $\tau_\beta = \frac{\partial}{\partial z_\beta}$ at 0. We extend $\tau_1, \ldots, \tau_{n-1}$ by their parallel transports along geodesics normal to ∂M so that they are smoothly defined in a neighborhood of 0. Denote $\tilde{u}_{\beta\bar{\gamma}} = u_{\tau_\beta \bar{\tau}_\gamma}$ and $\tilde{\mathfrak{g}}_{\beta\bar{\gamma}} = \tilde{u}_{\beta\bar{\gamma}} + \chi(\tau_\beta, \bar{\tau}_\gamma)$, $1 \leq \beta, \gamma \leq n - 1$, etc. On ∂M we have

$$\frac{n \hat{\chi}_u^{n-1}}{\psi (n - \alpha) \hat{\chi}_u^{n-\alpha-1} \wedge \hat{\omega}^\alpha} = \frac{C_n^\alpha}{\psi} \frac{S_{n-1}(\tilde{\mathfrak{g}}_{\beta\bar{\gamma}})}{S_{n-\alpha-1}(\tilde{\mathfrak{g}}_{\beta\bar{\gamma}})}. \tag{4.8}$$

Define, for a positive definite $(n-1)\times(n-1)$ Hermitian matrix $\{r_{\beta\bar\gamma}\}$,

$$G[r_{\beta\bar\gamma}] := \left(\frac{S_{n-1}(\lambda(r_{\beta\bar\gamma}))}{S_{n-\alpha-1}(\lambda(r_{\beta\bar\gamma}))}\right)^{\frac{1}{\alpha}},$$

where $\lambda(r_{\beta\bar\gamma})$ denotes the ordinary eigenvalues of $\{r_{\beta\bar\gamma}\}$ (with respect to the identity matrix I), and let

$$G_0^{\beta\bar\gamma} = \frac{\partial G}{\partial r_{\beta\bar\gamma}}[\mathfrak{g}_{\beta\bar\gamma}(0)].$$

Note that G is concave and homogeneous of degree one. Therefore,

$$\sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma} r_{\beta\bar\gamma} \geq G[r_{\beta\bar\gamma}] \tag{4.9}$$

for any $\{r_{\beta\bar\gamma}\}$. In particular, since $u_{\beta\bar\gamma}(0) = \underline{u}_{\beta\bar\gamma}(0) + (u-\underline{u})_{x_n}(0)\sigma_{\beta\bar\gamma}(0)$, we have

$$\begin{aligned} G[\mathfrak{g}_{\beta\bar\gamma}(0)] &= \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}\mathfrak{g}_{\beta\bar\gamma}(0) \\ &= \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}(\chi_{\beta\bar\gamma}(0) + \underline{u}_{\beta\bar\gamma}(0)) + (u-\underline{u})_{x_n}(0)\sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}\sigma_{\beta\bar\gamma}(0). \end{aligned} \tag{4.10}$$

We shall need the following elementary lemma.

Lemma 4.2. *Let*

$$A = \begin{bmatrix} B & C \\ \bar C' & a_{n\bar n} \end{bmatrix}$$

be a positive definite Hermitian matrix. Then

$$G^{\alpha}(B) \geq (1+c_0)\frac{S_n(\lambda(A))}{S_{n-\alpha}(\lambda(A))} \tag{4.11}$$

where $c_0 > 0$ depends on the lower and upper bounds of the eigenvalues of A.

Proof. It is straightforward to verify that

$$\begin{bmatrix} I & 0 \\ \bar C' B^{-1} & 1 \end{bmatrix} \begin{bmatrix} B & C \\ \bar C' & a_{n\bar n} \end{bmatrix} \begin{bmatrix} I & B^{-1}C \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & a_{n\bar n} - \bar C' B^{-1} C \end{bmatrix}.$$

So

$$\det A = (a_{n\bar n} - \bar C' B^{-1} C)\det B.$$

We now claim

$$S_{n-\alpha}(\lambda(A)) \geq (a_{n\bar n} - \bar C' B^{-1} C) S_{n-\alpha-1}(\lambda(B)) + S_{n-\alpha}(\lambda(B)).$$

To see this we can assume B is diagonal and consider a submatrix of A of the form

$$A_J = \begin{bmatrix} B_J & C_J \\ \bar{C}_J{}' & a_{n\bar{n}} \end{bmatrix}.$$

We have

$$\bar{C}' B^{-1} C \geq \bar{C}_J{}' B_J^{-1} C_J \geq 0$$

since B is positive definite and $\bar{C}'$ is the conjugate transpose of C. Therefore,

$$\det A_J = (a_{n\bar{n}} - \bar{C}_J{}' B_J^{-1} C_J) \det B_J \geq (a_{n\bar{n}} - \bar{C}' B^{-1} C) \det B_J.$$

The claim and (4.11) now follow easily. $\square$

We continue the proof of (1.4). Suppose that for some small $\theta_0 > 0$ to be determined later,

$$-\sum_{\beta,\gamma<n} (u - \underline{u})_{x_n}(0) G_0^{\beta\bar{\gamma}} \sigma_{\beta\bar{\gamma}}(0) \leq \theta_0 \sum_{\beta,\gamma<n} G_0^{\beta\bar{\gamma}} (\chi_{\beta\bar{\gamma}}(0) + \underline{u}_{\beta\bar{\gamma}}(0)).$$

Then,

$$\begin{aligned} G[\mathfrak{g}_{\beta\bar{\gamma}}(0)] &\geq (1-\theta_0) \sum_{\beta,\gamma<n} G_0^{\beta\bar{\gamma}} (\chi_{\beta\bar{\gamma}}(0) + \underline{u}_{\beta\bar{\gamma}}(0)) \\ &\geq (1-\theta_0)\, G[\chi_{\beta\bar{\gamma}}(0) + \underline{u}_{\beta\bar{\gamma}}(0)] \\ &\geq (1-\theta_0)(1+c_0) F(\chi_{\underline{u}}) \\ &\geq (1-\theta_0)(1+c_0) \left(\frac{\psi(0)}{C_n^\alpha}\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{4.12}$$

The second and fourth inequalities follow from (4.9) and (1.3), respectively, while the third from Lemma 4.2. Choosing θ_0 small enough, we obtain

$$m_0 = \frac{C_n^\alpha}{\psi(0)} G[\mathfrak{g}_{\beta\bar{\gamma}}(0)]^\alpha \geq 1 + \frac{\theta_0}{2}.$$

Suppose now that

$$-(u - \underline{u})_{x_n}(0) \sum_{\beta,\gamma<n} G_0^{\beta\bar{\gamma}} \sigma_{\beta\bar{\gamma}}(0) > \theta_0 \sum_{\beta,\gamma<n} G_0^{\beta\bar{\gamma}} (\chi_{\beta\bar{\gamma}}(0) + \underline{u}_{\beta\bar{\gamma}}(0)).$$

On ∂M, $\tilde{u}_{\beta\bar{\gamma}} = \tilde{\varphi}_{\beta\bar{\gamma}} + (u - \varphi)_\nu \tilde{\sigma}_{\beta\bar{\gamma}}$ where

$$\nu = \sum_{k=1}^{2n} \nu^k \frac{\partial}{\partial t_k}$$

is the interior unit normal vector field to ∂M. We have $|\nu^k| \leq C\rho$ for $k < 2n$ and $|(u-\varphi)_{t_k}| \leq C\rho$ since $\nu^k(0) = 0$ for $k < 2n$ and $u = \varphi$ on ∂M. Define

$$\begin{aligned} \Phi = & \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}(\tilde\chi_{\beta\bar\gamma} + \tilde\varphi_{\beta\bar\gamma}) + (u-\varphi)_{x_n}\nu^{2n} \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}\tilde\sigma_{\beta\bar\gamma} - \Big(\frac{m_0\psi}{C_n^\alpha}\Big)^{\frac{1}{\alpha}} \\ := & -(u-\varphi)_{x_n}\eta + Q \end{aligned} \tag{4.13}$$

where η and Q are smooth. Note that $\Phi(0) = 0$ and

$$\begin{aligned} \eta(0) = & -\nu^{2n}(0) \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}\sigma_{\beta\bar\gamma}(0) \\ & > \frac{\theta_0}{(u-\underline{u})_{x_n}(0)} \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}(\chi_{\beta\bar\gamma}(0) + \underline{u}_{\beta\bar\gamma}(0)) \\ & \geq \frac{\theta(1+\epsilon)\psi(0)}{C_n^\alpha(u-\underline{u})_{x_n}(0)} \geq c_2 > 0. \end{aligned} \tag{4.14}$$

On ∂M,

$$\begin{aligned} \Phi = & \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}\tilde{\mathfrak{g}}_{\beta\bar\gamma} - \sum_{k<2n}(u-\varphi)_{t_k}\nu^k \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}\tilde\sigma_{\beta\bar\gamma} - \Big(\frac{m_0\psi}{C_n^\alpha}\Big)^{\frac{1}{\alpha}} \\ \geq & \sum_{k<2n}(u-\varphi)_{t_k}\nu^k \sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}\tilde\sigma_{\beta\bar\gamma} \geq -C\rho^2 \end{aligned} \tag{4.15}$$

since by (4.9)

$$\sum_{\beta,\gamma<n} G_0^{\beta\bar\gamma}\tilde{\mathfrak{g}}_{\beta\bar\gamma} \geq G[\tilde{\mathfrak{g}}_{\beta\bar\gamma}] \geq \Big(\frac{m_0\psi}{C_n^\alpha}\Big)^{\frac{1}{\alpha}}.$$

We calculate

$$\begin{aligned} \sum_{i,j} F^{i\bar j}\Phi_{i\bar j} \leq & -\eta\sum_{i,j} F^{i\bar j}(u_{x_n})_{i\bar j} + C\sum_{i,j} F^{i\bar j} g_{i\bar j} \\ & + \sum_{i,j} F^{i\bar j}\,(u-\varphi)_{x_n z_i}(u-\varphi)_{x_n \bar z_j}. \end{aligned} \tag{4.16}$$

As in [3] (see also [18]),

$$\sum_{i,j} F^{i\bar j}(u-\varphi)_{x_n z_i}(u-\varphi)_{x_n\bar z_j} \leq \sum_{i,j} F^{i\bar j}(u-\varphi)_{y_n z_i}(u-\varphi)_{y_n\bar z_j} + C\sum_{i,j} F^{i\bar j} g_{i\bar j} + C.$$

On the other hand, differentiating equation (2.9) with respect to x_n, we see that

$$-\sum_{i,j} F^{i\bar j}(u_{x_n})_{i\bar j} \leq 2\Big|\sum_{i,j,l} F^{i\bar j}\mathfrak{g}_{i\bar l}\overline{\Gamma^l_{nj}}\Big| + C\sum_{i,j} F^{i\bar j} g_{i\bar j} + C. \tag{4.17}$$

At a fixed point choose a unitary $A = \{a_{ij}\}_{n\times n}$ which diagonalizes $\{\mathfrak{g}_{i\bar{j}}\}$. We have

$$\begin{aligned}\sum_{i,j,l} F^{i\bar{j}}\mathfrak{g}_{i\bar{l}}\overline{\Gamma^l_{nj}} =& \sum_{i,j,l,s,t,p,q} a^{is} f_s \delta_{st}\bar{a}^{jt} a_{ip}\lambda_p\delta_{pq}\bar{a}_{lq}\overline{\Gamma^l_{nj}} \\ =& \sum_q f_q\lambda_q \sum_{j,l} \bar{a}^{jq}\bar{a}_{lq}\overline{\Gamma^l_{nj}} \leq C\psi.\end{aligned} \tag{4.18}$$

Therefore,

$$-\sum_{i,j} F^{i\bar{j}}(u_{x_n})_{i\bar{j}} \leq C\sum_{i,j} F^{i\bar{j}} g_{i\bar{j}} + C. \tag{4.19}$$

Applying Lemma 4.1 we derive

$$\sum_{i,j} F^{i\bar{j}}(Av + B\rho^2 + \Phi - |(u-\varphi)_{y_n}|^2)_{i\bar{j}} \leq 0 \text{ in } M\cap B_\delta(0)$$

and $Av + B\rho^2 + \Phi - |(u-\varphi)_{y_n}|^2 \geq 0$ on $\partial(M\cap B_\delta(0))$ when $A \gg B \gg 1$. By the maximum principle, $Av + B\rho^2 + \Phi - |(u-\varphi)_{y_n}|^2 \geq 0$ in $M\cap B_\delta(0)$, and therefore $\Phi_{x_n}(0) \geq -C$. This gives

$$u_{n\bar{n}}(0) \leq C.$$

We now have positive lower and upper bounds for all eigenvalues of $\{\mathfrak{g}_{i\bar{j}}(0)\}$. By Lemma 4.2,

$$G[\mathfrak{g}_{\beta\bar{\gamma}}(0))] \geq (1+c_0)F(\mathfrak{g}_{i\bar{j}}(0))$$

for some $c_0 > 0$. It follows that

$$m_0 = \frac{C_n^\alpha}{\psi(0)} G[\mathfrak{g}_{\beta\bar{\gamma}}(0))] \geq 1 + c_0.$$

The proof of (1.4) is therefore complete.

5. The second order estimates

Proposition 5.1. *Suppose $\chi \in \mathscr{C}_\alpha(\omega)$ and let $u \in C^4(M)\cap C^2(\bar{M})$ be a solution of equation* (1.1)*. Then there is a uniform constant $C > 0$ such that*

$$\sup_{\overline{M}} \Delta u \leq C(1 + \sup_{\partial M} \Delta u). \tag{5.1}$$

Proof. Let ϕ be a function to be determined later and assume that we^{ϕ} reaches its maximum at some point $p \in M$ where $w = \Delta u + tr\chi$. Choose local coordinates around p such that $g_{i\bar{j}}(p) = \delta_{ij}$ and $\mathfrak{g}_{ij}$ is diagonal. At p we have

$$\frac{\partial_l w}{w} + \partial_l \phi = 0, \quad \frac{\bar{\partial}_l w}{w} + \bar{\partial}_l \phi = 0 \tag{5.2}$$

and

$$\frac{\bar{\partial}_l \partial_l w}{w} - \frac{\bar{\partial}_l w \partial_l w}{w^2} + \bar{\partial}_l \partial_l \phi \leq 0. \tag{5.3}$$

By (5.2) and Schwarz inequality,

$$\begin{aligned} \big|\partial_l w\big|^2 = \Big|\sum_i \mathfrak{g}_{i\bar{i}l}\Big|^2 &= \Big|\sum_i (\mathfrak{g}_{l\bar{i}i} - T^i_{li}\mathfrak{g}_{i\bar{i}}) + \lambda_l\Big|^2 \\ &\leq w \sum_i \mathfrak{g}^{i\bar{i}} \big|\mathfrak{g}_{l\bar{i}i} - T^i_{li}\mathfrak{g}_{i\bar{i}}\big|^2 - 2w\mathfrak{Re}\{\phi_l \bar{\lambda}_l\} - \big|\lambda_l\big|^2 \end{aligned} \tag{5.4}$$

where

$$\lambda_l = \sum_i \Big(\chi_{i\bar{i}l} - \chi_{l\bar{i}i} + \sum_j T^j_{li}\chi_{j\bar{i}}\Big).$$

Next, by (2.6) and (2.14),

$$\begin{aligned} \sum_l S_{\alpha-1;l}(\mathfrak{g}^{l\bar{l}})^2 \bar{\partial}_l \partial_l w &= \sum_{i,l} S_{\alpha-1;l}(\mathfrak{g}^{l\bar{l}})^2 \mathfrak{g}_{i\bar{i}l\bar{l}} \\ &\geq \sum_{i,l} S_{\alpha-1;l}(\mathfrak{g}^{l\bar{l}})^2 \mathfrak{g}_{l\bar{l}i\bar{i}} - 2\sum_{i,j,l} S_{\alpha-1;l}(\mathfrak{g}^{l\bar{l}})^2 \mathfrak{Re}\{\overline{T^j_{li}}\mathfrak{g}_{l\bar{j}i}\} \\ &\quad + \sum_{i,j,l} S_{\alpha-1;l}(\mathfrak{g}^{l\bar{l}})^2 T^j_{li}\overline{T^j_{li}}\mathfrak{g}_{j\bar{j}} - Cw\sum_l S_{\alpha-1;l}(\mathfrak{g}^{l\bar{l}})^2 \\ &\geq \sum_{i,j,l} S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2 \mathfrak{g}^{j\bar{j}} \big|\mathfrak{g}_{i\bar{j}l} - T^j_{il}\mathfrak{g}_{j\bar{j}}\big|^2 \\ &\quad - Cw\sum_l S_{\alpha-1;l}(\mathfrak{g}^{l\bar{l}})^2 - C. \end{aligned} \tag{5.5}$$

It follows from (5.3), (5.4) and (5.5) that

$$\begin{aligned} 0 \geq w\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2 \phi_{i\bar{i}} &+ 2\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2 \mathfrak{Re}\{\phi_i \bar{\lambda}_i\} \\ - Cw\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2 - C.& \end{aligned} \tag{5.6}$$

Let $\phi = e^{A\eta}$ with $\eta = \underline{u} - u + \sup_M(u - \underline{u})$, where $\underline{u} \in C^2(\bar{M})$ satisfies $\chi_{\underline{u}} > 0$ and (2.15), and A is a positive constant to be determined. So

$$\phi_i = A\phi\eta_i, \quad \phi_{i\bar{i}} = A\phi\eta_{i\bar{i}} + A^2\phi|\eta_i|^2.$$

Applying Schwarz inequality again,

$$\begin{aligned} 2\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2\mathfrak{Re}\{\phi_i\bar{\lambda}_i\} &= 2A\phi\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2\mathfrak{Re}\{\eta_i\bar{\lambda}_i\} \\ &\geq -wA^2\phi\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2|\eta_i|^2 - \frac{C\phi}{w}\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2. \end{aligned} \tag{5.7}$$

Finally, by (5.6) and (5.7),

$$wA\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2\eta_{i\bar{i}} \leq \frac{C}{\phi} + C\Big(\frac{1}{w} + \frac{w}{\phi}\Big)\sum_i S_{\alpha-1;i}(\mathfrak{g}^{i\bar{i}})^2. \tag{5.8}$$

From Lemma 2.1, this gives a bound $w \leq C$ at p for A sufficiently large. $\square$

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